# Planar subdivisions, half-edge datastructure, Euler's formula

## 1 Planar subdivisions

We'll be interested in representations and properties of planar subdivisions. Planar subdivision can be defined as a finite set of edges in the plane such that any two edges are either disjoint or meet at a common endpoint. We'll focus on 'edges' being straight lines below but in fact they need not be straight lines: the datastructures work and the properties that we'll discuss carry over to subdivisions based on curved lines. An example of a planar subdivision is shown in Figure 1.

By a face of a planar subdivision we mean a connected component in the complement of the union of its edges. For example, the subdivision in Figure 1 has 6 faces. One of them is unbounded. In general, every planar subdivision has exactly one unbounded face.

Planar subdivisions are convenient in many contexts. In cartography, they can be used to represent maps (e.g. political maps where faces are states or counties). One can also consider subdivisions on 3D surfaces. In fact, representations of surfaces based on polygons can be thought of as subdivisions of these surfaces into these polygons (which play the role of faces).

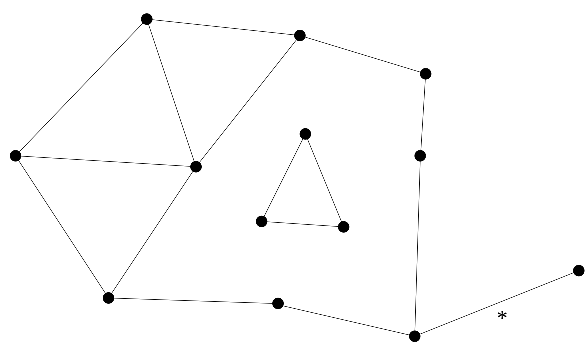
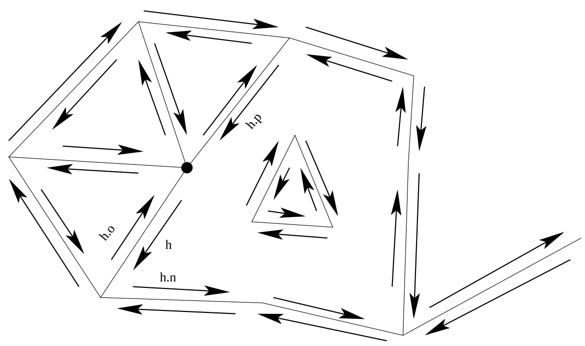



Figure 1: A planar subdivision.

## 2 Half-edge data structure

Planar subdivisions can be represented using half-edge datastructure. Its basic building block can be viewed as an arrow running along an edge. The way the arrows are constructed is as follows. For each edge in the subdivision draw arrows on its both sides, each of them in such a way that the edge is on its right. This leads to an arrangement of arrows like that shown in the Figure below.




Thus, all edges have two arrows along them, running in different directions. This is where the name half-edge comes from (edge=two arrows, so an arrow=edge/2). Each of the half edges $h$ is stored as a record, having the following fields (see the figure above):

**1.** the reference to the next half edge, $h.n$

**2.** the reference to the previous half-edge, $h.p$

**3.** the reference to the opposite half-edge, $h.o$

**4.** the reference to the starting vertex, $h.s$

**5.** the reference to its face, $h.f$.

Except for the half-edge records, there are vertex records as well. A vertex record stores vertex data (like coordinates, if the datastructure is to represent a 3D surface then perhaps color, normals and texture coordinates) and a pointer to one of the half edges out of it (any will do). This pointer, together with all other components of the half edge data structure, allows to efficiently find all vertices, faces or half edges incident to any specified vertex.

Finally, we also store face records. Apart from face attributes they contain a pointer to a half edge on the face's outer bounding loops. We also store a list of pointers to half edges on its inner bounding loops (one per loop). These pointers allow to efficiently execute neighborhood queries for the faces.

Half edge data structure allows to efficiently (i.e. in time proportional to the local complexity of the subdivision) answer neighborhood queries such as these:

**1.** output all vertices adjacent to a given one

**2.** output all faces adjacent to a given face across an edge

**3.** output all half edges bounding a face.

## 3 Euler's formula for a planar subdivisions

Let $F$, $E$, $V$ and $C$ be the number of faces, edges, vertices and connected components of a subdivision (respectively). By number of connected components we mean the number of connected components of the graph formed by vertices and edges in the subdivision in the usual graph theoretic sense. For example, the subdivision in Figure 1 has 2 connected components.

The Euler formula states that for any planar subdivision

$$F - E + V = 1 + C.$$

The left hand side (which can be thought of as alternating sum of numbers of mesh elements of different dimensions) is often called the Euler characteristics.

The proof of the Euler's formula is simple: take any planar subdivision and remove one edge. There are two possibilities:

**1.** The removal increases the number of connected components $C$ by one (e.g. edge marked with '*' in Figure 1)

**2.** $C$ does not change as a result of the edge removal.

In the first case, the numbers of faces and vertices do not change and therefore both the left hand side and the right hand side of the Euler formula increase by one. In the second case, the numbers of both edges and faces decrease by one and the number of vertices also stays fixed. Thus, both sides of the Euler formula do not change.

This means that the Euler formula holds for a subdivision if and only if it holds for a subdivision with one edge removed. By induction, it holds for a subdivision iff it holds for the subdivision with all edges removed. But after all edges are removed, what we are left with is $n$ separate vertices. Thus, $C = n$, $V = n$, $E$=0, $F$=1 and the Euler's formula is obviously satisfied.

## 4 Euler's formula for triangulated surfaces

Euler's formula holds not only for planar subdivisions: it is true for subdivisions of spheres (i.e. graphs drawn on spheres) as well. In fact subdivisions of spheres induce planar subdivisions with the same numbers of faces, edges, vertices and components as follows. Put the sphere with the subdivision on a table, making sure its highest point is not on an edge of a subdivision. Imagine the sphere is made of a transparent material and the subdivision edges are drawn on it using an opaque ink. Put a point light source at the highest point of the sphere. The shadows of the edges of the subdivision on the sphere form curves on the table that we'll treat as edges of the corresponding planar subdivision. Since the Euler's formula holds for the planar subdivision, it also has to hold for the subdivision on the sphere.

If all faces of the subdivision on the sphere have three edges and the graph is connected ($C = 2$) then Euler formula says that $F - E + V = 2$. Notice that in this case $3F = 2E$: this is because every triangle has three edges and every edge has exactly two incident triangles (imagine you go over faces and, for each face, count its edges; after all faces are processed, you have counted every edge exactly twice; on the other hand, you counted to $3T$). Thus, in this case the Euler's formula reduces to $F + 4 = 2V$ or $3V = 6 + E$. In particular, this means that for a fine triangulation of a sphere, there are roughly twice as many triangles as vertices (first equation). Moreover, the average vertex degree in such a triangulation is close to 6 (this follows from the second equation: go over vertices and, for each vertex, count all edges out of it; you'll count to the sum of all vertex degrees while counting every edge twice; and from the second equation, twice the number of edges is (up to a constant) equal to six times the number of vertices).

The Euler formula as discussed above can be generalized from triangulations of spheres to triangulations of surfaces *homeomorphic* to the sphere. A homeomorphism is a continuous, function that is invertible and has a continuous inverse. Intuitively, if a surface can be continuously (without tearing, cutting or gluing) deformed to a sphere, it is homeomorphic to a sphere.

Euler's formula can be used to recognize the topological type of a 3D surface. A classification theorem for orientable surfaces says that any orientable surface is homeomorphic to a sphere with a some number of *handles*. Handles are hard to define formally, so let's just say that torus is a sphere with one handle, double torus is a sphere with two handles (Figure 2). In this case by *genus* we mean the number of handles. Sphere has genus zero.

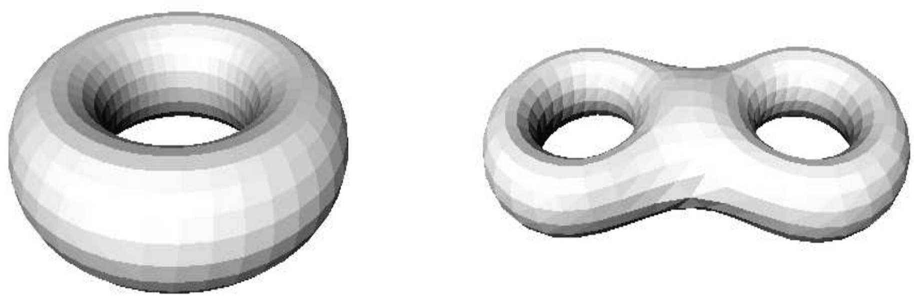


Figure 2: Torus (genus 1) and double torus (genus 2): sphere with one or two handles.

Any triangulation of a surface of genus $g$ has to fulfill Euler's formula

$$F - E + V = 2 - 2g.$$

In fact, it is not hard to deduce this formula from the Euler's formula for triangulations of a sphere. What do we need to get rid of a handle? For example, we could find a loop running around the torus, cut it along that loop and then fill the resulting holes (Figure 3) by triangulating them. How does this change the numner of edges, faces and vertices? Assuming the loop along which the mesh was cut has $n$ vertices, the number of vertices increases by $n$ (each vertex on the loop is split into two when cutting), the number of triangles increases by $2(n-2)$ (there are two loops to triangulate, each having $n$ edges – it takes $n-3$ triangles to triangulate an $n$-gon) and the number of edges increases by $n+2(n-3)$. Therefore, the Euler characteristic $F-E+V$ increases by 2. Since genus-$g$ surface requires $g$ such edits to remove all handles, its Euler characteristic is $2-2g$.

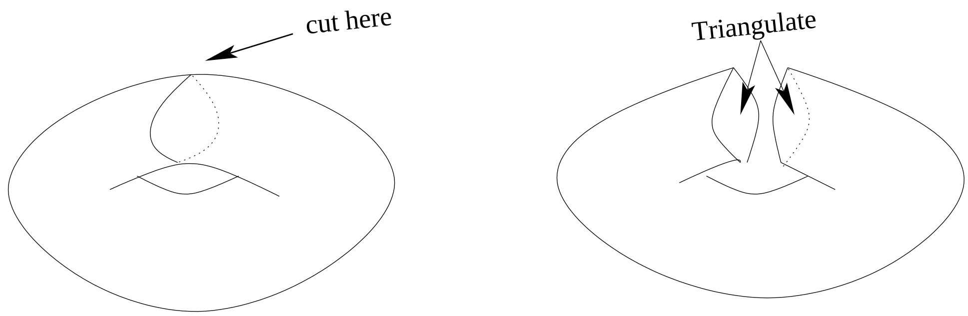



Figure 3: Instructions for handle removal.