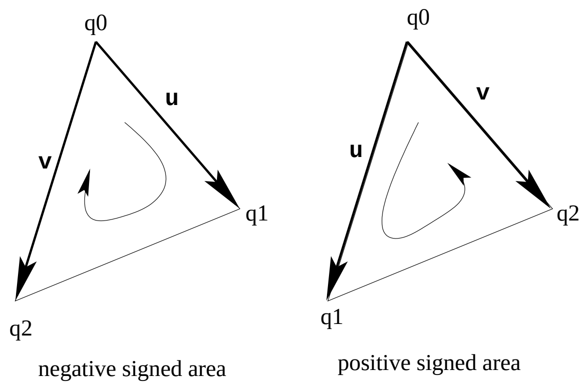

Figure 1: Orientation and the sign of signed area.

Computing areas in 2D and volumes in 3D

Ability to compute or estimate area/volume is useful in a number of geometry processing algorithms, including simplification.

1 2D case

1.1 Triangle

Consider a triangle with vertices $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$. Its area is obviously the same as the area of the 3D triangle with vertices $q_0 = (x_0, y_0, 0)$, $q_1 = (x_1, y_1, 0)$ and $q_2 = (x_2, y_2, 0)$. Now, the area of this triangle can be computed as half of the length of the cross product of vectors running along its edges ($\vec{u}$ from q_0 to q_1 and $\vec{v}$ from q_0 to q_2). Notice that these vectors live in the xy-plane. Therefore, the first two coordinates of the cross product are zero (it has to be perpendicular to the xy-plane and therefore parallel to the z-axis) and its magnitude is equal to the absolute value of the z-coordinate. The z-coordinate is equal to $\det \begin{bmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{bmatrix}$. We sum up with the following formula for the area of the triangle $\Delta p_0 p_1 p_2$:

$$\text{Area}(\Delta p_0 p_1 p_2) = \frac{1}{2} |\det \begin{bmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{bmatrix}| = \frac{1}{2} |\det \begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}|.$$

Notice that the sign of the determinant under the absolute value depends on the orientation of the triangle defined by the ordering of its vertices. If the triangle is oriented clockwise, the area is negative, otherwise it is positive (Figure 1). In what follows, we'll call a half of this determinant the *signed area*.

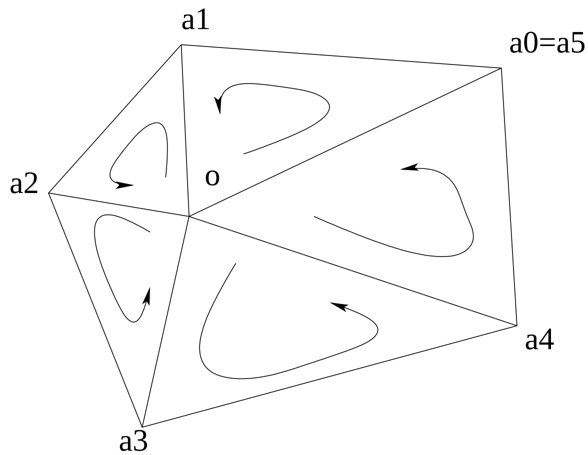

Figure 2: Area of convex polygon, case 1: o inside.

1.2 Convex polygon

Let $a_0, a_1, \dots a_{n-1}$ be vertices of a convex polygon in the counterclockwise order. In what follows we mean a_0 whenever we write a_n. In order to compute the area of the polygon, choose an arbitrary point o. To start with, assume o is inside the polygon (Figure 2). Then, every triangle $\Delta(o, a_i, a_{i+1})$ for $i = 0, 1, \dots, n-1$ is oriented counterclockwise and therefore and therefore its signed area is positive. Hence the area of the polygon can be computed by simply summing the signed areas. If the vertices were ordered clockwise, all signed areas would be negative and therefore the area can be computed as the absolute value of the sum of all signed areas. Interestingly, this is still true even if o is outside the polygon, as illustrated in figure 3. Thus, area of the polygon is equal to

$$|\sum_{i=0}^{n-1} \frac{1}{2}\det\begin{bmatrix} 1 & o_x & o_y \\ 1 & a_{i,x} & a_{i,y} \\ 1 & a_{i+1,x} & a_{i+1,y} \end{bmatrix}| \qquad (1)$$

This formula works for any convex polygon, if only its vertices are numbered in order (either clockwise or counterclockwise).

1.3 Any polygon, convex or not

It turns out that formula 1 works also if the polygon is not convex. This is illustrated in Figure 4.

For a point p in the plane, let $n(p)$ be the number of blue triangles containing p and $m(p)$ - the number of red triangles containing p. Notice that if p is not on an edge connecting a vertex of the polygon to o or on an edge of the polygon, two cases are possible:

1. $n(p) = m(p)$. This happens if p is outside the polygon.

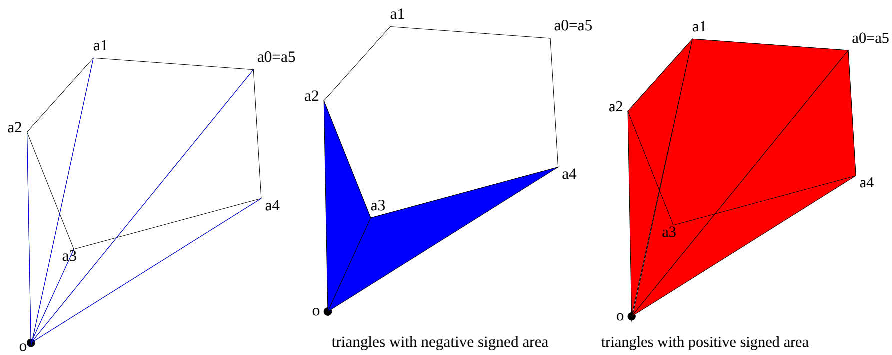

Figure 3: Area of convex polygon, case 2: o outside. Left: setup, Center: triangles with negative signed area, right: triangles with positive signed area. The red area minus the blue area is exactly equal to the area of the polygon.

2. $n(p) = m(p) \pm 1$ if p is inside the polygon. Whether we have to use $+$ or $-$ depends on whether the polygon's vertices come in clockwise or counterclockwise order.

Sloppily speaking, in formula 1, points outside of the polygon do not contribute to the area or to the sum of signed areas of triangles defined by o and pairs of consecutive vertices (the expression under the absolute value in (1)). Points inside are counted, either as 'negative' or as 'positive', depending on the orientation of the polygon. Therefore, the sum of signed areas of all triangles defined by o and the edges of the polygon is equal to either positive or negative area of the polygon. Note that this is remotely related to the 'zfail' version of the shadow volume algorithm: take a p and shoot a ray from o toward p. Past p, this ray is going to intersect some (or none) edges of the polygon. If p is outside the polygon, the ray has to exit from the polygon the same number of times as it enters it. Edges through which it enters together with o define triangles containing p oriented one way. Edges through each it exits with o define triangles containing p oriented the other way. The numbers of triangles with the two different orientations have to be the same. If p is inside the polygon, number of triangles with oriented the first way exceeds the number of triangles oriented the other way by 1. One can conclude that $m(p) - n(p)$ is zero outside the polygon and is either equal to 1 or -1 inside.

2 3D case

All the formulas carry over to the 3D case. Here is how.

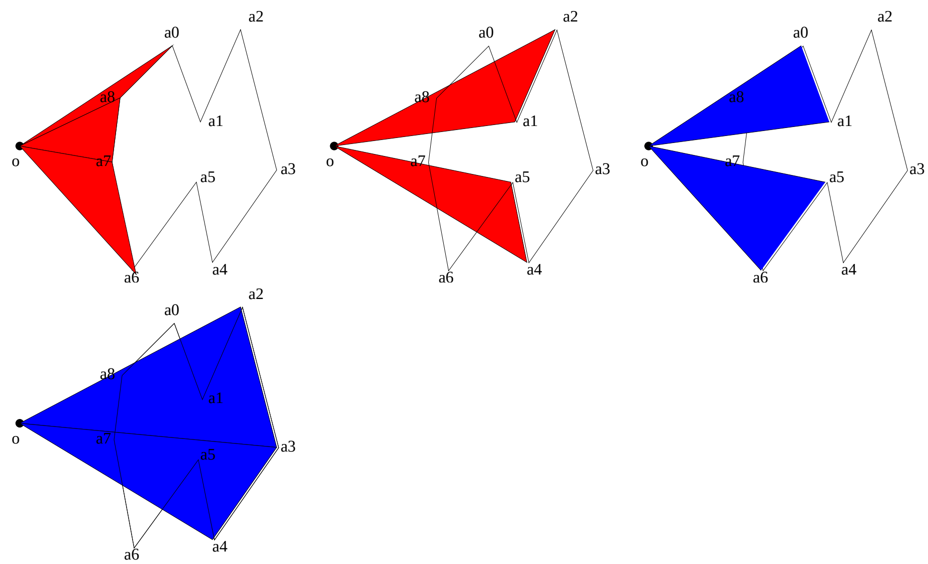

Figure 4: All triangles formed by o and two consecutive vertices of the polygon with positive signed area (red) and negative signed area (blue). Notice that the total area of red triangles minus the total area of blue triangles is equal to the negative area of the polygon itself. This means that formula (1) can be used to compute the polygon's area.

2.1 Tetrahedron

The volume of a tetrahedron $T(a, b, c, d)$ with vertices a, b, c and d is given by

$$\text{vol } T(a,b,c,d) = \frac{1}{6}|\vec{ab} \cdot (\vec{ac} \times \vec{ad})| = \frac{1}{6}|\det \begin{bmatrix} b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \\ d_x - a_x & d_y - a_y & d_z - a_z \end{bmatrix}| =$$

$$= \frac{1}{6}|\det \begin{bmatrix} 1 & a_x & a_y & a_z \\ 1 & b_x & b_y & b_z \\ 1 & c_x & c_y & c_z \\ 1 & d_x & d_y & d_z \end{bmatrix}|$$

This is because the cross product $\vec{ac} \times \vec{ad}$ is a vector perpendicular to the plane spanned by $\vec{ac}$ and $\vec{ad}$ and its magnitude is equal to half of the area of the triangle $\Delta(a, c, d)$ (which we will treat as the 'base' of the tetrahedron). Scale the cross product to a unit vector N. Notice that the absolute value of the dot product $N \cdot \vec{ab}$ is equal to the height of the tetrahedron (it's a projection of $\vec{ab}$ onto a direction perpendicular to the base). Using the formula from high school, volume is equal to one-third of base area times height, i.e.

$$|\frac{1}{6} * \text{area}(\Delta(a,c,d)) * N \cdot \vec{ab}| = |\frac{1}{6}(\vec{ac} \times \vec{ad}) \cdot \vec{ab}|.$$

Just as in the 2D case, we'll think of the expression under the absolute value as the signed area of the tetrahedron. Note that its sign depends on whether the triangle $\Delta(b, c, d)$ appears oriented clockwise or counterclockwise when looked at from a.

2.2 Volume enclosed by any triangular mesh

Essentially the same formula as in the 2D case works in 3D. To compute volume enclosed by a triangle mesh, orient all triangles of the mesh consistently. Select any point o and use the formula:

$$|\sum_{T(a,b,c):\text{triangle in the mesh}} \frac{1}{6}\det \begin{bmatrix} 1 & a_x & a_y & a_z \\ 1 & b_x & b_y & b_z \\ 1 & c_x & c_y & c_z \\ 1 & d_x & d_y & d_z \end{bmatrix}|$$

The proof is essentially the same: count the number of tetrahedra with negative and positive signed volume containing a point p. Assume p is not on any boundary of a tetrahedron formed by o and a triangle in the mesh. The number of negative tetrahedra will be the same as the number of positive tetrahedra if p is not enclosed by the mesh and the two numbers will differ by 1 otherwise (cf 2D case and z-fail shadow volume algorithm). Note that it is important that we use consistently oriented triangles: basically, this ensures that the positive and negative signed areas cancel properly to produce the correct result.